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Parametrically excited standing edge waves

By JOHN MILES
Institute of Geophysics and Planetary Physics, University of California, San Diego, La Jolla, CA 92093, USA

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The resonant excitation of weakly nonlinear, standing edge waves of frequency ω and longshore wavenumber k by a normally incident, non-breaking gravity wave of frequency 2ω and shoreline amplitude a on a bottom that descends smoothly from a shoreline depth of zero and slope σ to an offshore depth h_∞ is calculated for $ka \ll \sigma \ll kh_\infty \ll 1$. The analysis generalizes hose of Guza & Bowen (1976) and Rockliff (1978), which assume uniform slope and perfect reflection, and culminates in a pair of evolution equations for the slowly varying, quadrature amplitudes (or, equivalently, amplitude and phase) of the edge wave. Weak, linear damping (which implies imperfect reflection) is incorporated, and the resulting fixed points and bifurcation points of the evolution equations are determined. It is shown that the solution for prescribed initial conditions must tend to one of the stable fixed points, which correspond to an edge-wave amplitude of either zero or $O((\sigma a/k)^{\frac{1}{2}})$, depending on whether the damping exceeds or is inferior to a certain critical value. The restriction $\sigma \ll kh_\infty$ is relaxed for the special depth profile $h/h_\infty = 1-\exp(-\sigma x/h_\infty)$, for which the inviscid, shallow-water equations admit exact solutions. These solutions serve to validate the asymptotic ($\sigma/kh \downarrow 0$) approximations for arbitrary depth profiles.

1. Introduction

I consider here the subharmonic excitation of a standing edge wave of longshore wavenumber k and natural frequency ω_0 by a normally incident gravity wave of frequency 2ω and shoreline amplitude a on a gently sloping bottom with the smooth depth profile

$$h = h_\infty \hat{h}(\sigma x/h_\infty) \sim \begin{matrix} \sigma x \ (\sigma x \ll h_\infty) \\ h_\infty \ (\sigma x \gg h_\infty) \end{matrix} \quad (0 < x < \infty), \tag{1.1}$$

on the assumptions that

$$\epsilon \equiv \frac{ka}{\sigma} \ll 1, \quad \mu \equiv \frac{\sigma}{kh_\infty} \ll 1, \quad kh_\infty \ll 1, \tag{1.2a–c}$$

$$\lambda \equiv \frac{\omega^2}{\sigma g k} = O(1), \quad \beta \equiv \frac{\omega^2-\omega_0^2}{2\epsilon\omega^2} \equiv \frac{\lambda-\lambda_0}{2\epsilon\lambda} = O(1). \tag{1.3a, b}$$

The dimensionless parameters ϵ, μ, kh_∞, λ and β are measures of nonlinearity, beach curvature (i.e. the decay of the slope from σ to 0), shallowness, frequency (squared) and proximity to resonance. The natural scale for h in the neighbourhood of the shore is σ/k, whence it is expedient to supplement (1.1) with

$$\hbar \equiv \sigma^{-1}kh = \mu^{-1}\hat{h}(\mu kx). \tag{1.4}$$

The problem posed in the preceding paragraph has been analysed for perfect reflection (of the incident wave) from a uniformly sloping beach ($h = \sigma x$) by Guza & Bowen (1976) and Rockliff (1978) using the shallow-water equations and by Minzoni & Whitham (1977) using the full equations of fluid motion with $\sigma = \tan(\pi/2N)$ for integer N. (The shallow-water approximation is manifestly inconsistent with $h = \sigma x \uparrow \infty$ as $x \uparrow \infty$, but Minzoni & Whitham find that it yields the correct results in the neighbourhood of the shore if $\sigma \ll 1$.) I revisit this problem (i) to accommodate an arbitrary depth profile for $0 < \mu \ll 1$, (ii) to obtain the solution for Ball's (1967) exponential profile for arbitrary $\mu > 0$, (iii) to determine the effects of imperfect reflection, and (iv) to determine the structure of the evolution equations and the stability of their solutions.

I delay consideration of imperfect reflection until §7 and first posit the shoreline and offshore displacements of the basic (incident plus reflected) standing wave in the forms

$$\zeta_s = a \sin 2\omega t \quad (x = 0) \tag{1.5}$$

and

$$\zeta_s \sim a_\infty \cos[k_\infty(x-l)] \sin 2\omega t \quad (k_\infty x \uparrow \infty), \quad k_\infty = 2\omega(gh_\infty)^{-\frac{1}{2}}, \qquad (1.6a, b)$$

where a_∞/a and $k_\infty l$ are determined by the solution of the inviscid reflection problem.

Guided by analogy with the problem of standing cross-waves in a wave tank (Miles 1988), I pose the edge-wave displacement at the shoreline in the form

$$\zeta_e = \epsilon^{-\frac{1}{2}} a[p(\tau) \cos \omega t + q(\tau) \sin \omega t] \cos ky[1+O(\epsilon^{\frac{1}{2}})] \quad (x = 0), \tag{1.7}$$

where p and q are slowly varying amplitudes and $\tau \equiv \epsilon\omega t$ is a slow time. In the cross-wave problem, absent dissipation, p and q are canonical variables that satisfy Hamiltonian equations (cf. (1.9) below). In the present problem energy is lost through both friction and radiation (Guza & Bowen 1976), and the resulting evolution equations are of the form

$$\dot{p} = -[\alpha + P + R(p^2+q^2)]p - [\beta + Q + S(p^2+q^2)]q, \tag{1.8a}$$

$$\dot{q} = -[\alpha - P + R(p^2+q^2)]q + [\beta - Q + S(p^2+q^2)]p, \tag{1.8b}$$

in which $\cdot \equiv \mathrm{d}/\mathrm{d}\tau$, α is a measure of linear damping (see §7), β is the tuning parameter (1.3b), P, R and S are measures of parametric excitation, radiation damping and self-interaction of the edge wave (see §4), and Q is a measure of imperfect reflection (see §7). I prove (in §4) that the solution of (1.8) must tend (as $\tau \uparrow \infty$) to a stable fixed point in the (p, q)-plane. If $\alpha > (P^2+Q^2)^{\frac{1}{2}}$ $p = q = 0$ is the only fixed point and edge waves decay. If $\alpha < (P^2+Q^2)^{\frac{1}{2}}$ there are symmetry-breaking bifurcations at $\beta = \pm(P^2+Q^2-\alpha^2)^{\frac{1}{2}}$, and the fixed point at $p = q = 0$ loses stability to either of a pair of finite-amplitude fixed points that represent standing edge waves.

It is worth noting that the evolution equations (1.8) are of the quasi-canonical form

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = -\frac{\partial D}{\partial p} - \frac{\partial H}{\partial q}, \quad \frac{\mathrm{d}q}{\mathrm{d}\tau} = -\frac{\partial D}{\partial q} + \frac{\partial H}{\partial p}, \qquad (1.9a, b)$$

where D and H are the dissipation and Hamiltonian functions

$$D = \tfrac{1}{2}\alpha(p^2+q^2) + \tfrac{1}{4}R(p^2+q^2)^2, \quad H = \tfrac{1}{2}\beta(p^2+q^2) + Ppq + \tfrac{1}{2}Q(q^2-p^2) + \tfrac{1}{4}S(p^2+q^2)^2. \qquad (1.10a, b)$$

I proceed as follows. In §2, I formulate the inviscid problem and posit an asymptotic expansion for the dimensionless velocity potential in powers of $\epsilon^{\frac{1}{2}}$, in which the dominant component, $\epsilon^{\frac{1}{2}}\phi_1$, is an eigensolution ($\lambda = \lambda_0$) of the linear edge-wave problem, the second-order components, $\epsilon\phi_0$ and $\epsilon\phi_2$, represent the basic wave and the second-order interaction of the edge wave, and the third-order component, $\epsilon^{\frac{3}{2}}\phi_3$, is driven by resonant detuning ($\propto \lambda - \lambda_0 = 2\epsilon\beta\lambda$) and the slow ($\tau$) variation of $\epsilon^{\frac{1}{2}}\phi$, the quadratic interaction between $\epsilon^{\frac{1}{2}}\phi_1$ and $\epsilon\phi_0$, and the cubic self-interaction of $\epsilon^{\frac{1}{2}}\phi_1$. This leads to a sequence of partial differential equations of the form $\mathscr{L}\phi_0 = \mathscr{L}\phi_1 = 0$ and $\mathscr{L}\phi_n = f_n (n = 2, 3, \ldots)$, where $\mathscr{L}$ is a second-order operator, f_2 is a functional of ϕ_1, and f_3 is a functional of ϕ_0, ϕ_1 and ϕ_2.

The solutions of $\mathscr{L}\phi_0 = \mathscr{L}\phi_1 = 0$ may be obtained through separation of variables. The functional f_2, which is quadratic in ϕ_1 and its derivatives (which are linear functionals of the slowly varying amplitudes $p(\tau)$ and $q(\tau)$) is orthogonal to the eigensolutions (which are sinusoidal in ωt and ky) of the operator $\mathscr{L}$, by virtue of which $\mathscr{L}\phi_2 = f_2$ admits a Fourier-transform solution for unrestricted $p(\tau)$ and $q(\tau)$. I obtain a formal representation of this solution in §3.

The functional f_3, in contrast to f_2, contains (for unrestricted p and q) Fourier components that resonate with the eigensolutions of $\mathscr{L}\phi$, and $\mathscr{L}\phi_3 = f_3$ is solvable if and only if p and q are chosen to render f_3 orthogonal to these eigensolutions (the Fredholm alternative). I invoke these solvability conditions in §4 to obtain the evolution equations (1.8), in which the parameters P, R and S appear as functionals of the kx-dependent factors of ϕ_0 and ϕ_1, $Z(kx)$ and $E(kx)$, respectively. I also incorporate the damping parameter α at this stage, but neglect Q, which proves to be $O(\alpha\epsilon)$ (see §7).

Further progress requires the determination of Z and E for specific profiles. I consider an exponential profile, for which Z and E are hypergeometric functions (Ball 1967; Miles 1990*b*), in §5 and Appendices A and B. These solutions are valid for all $\mu > 0$, but they are of special interest for $\mu \ll 1$, in which domain their power-series developments provide a test of the asymptotic ($\mu \downarrow 0$) results for other profiles.

In §6, I develop the asymptotic solution for a profile of the form (1.1) and show that P, R and S are given within $1 + O(\mu)$ by the corresponding results for a uniformly sloping beach. I also obtain the $O(\mu)$ corrections to P, R and S for finite curvature.

The development in §§2–6 neglects viscous and capillary effects (except as they are anticipated in §4); I consider these effects in §7. The parameter α, which represents viscous dissipation of the edge wave, and the correction of ω_0 for viscous dispersion may be obtained through a boundary-layer approximation (cf. Guza & Bowen 1976) despite the violation of the boundary-layer approximation near $h = 0$. The primary effect of viscosity on the basic wave is to render the reflection imperfect. The introduction of viscosity, absent capillarity, renders the singularity at $h = 0$ irregular (it is regular for the inviscid problem), and it is necessary to invoke capillarity to obtain a regular singularity. I have considered this viscous reflection problem elsewhere (Miles 1990*b*) and merely state the required results in §7.

The present results appear to be adequate for the analytical treatment of the edge-wave problem (although it may be expedient to evaluate the damping parameter α, and perhaps also Q, through direct measurement) on real or laboratory beaches in the absence of breaking; however, it must be emphasized that the conditions under which breaking is absent and standing edge waves are realized on real beaches are rather special.

2. Perturbation expansion

Invoking $kh \ll 1$ and assuming irrotational flow, we have the shallow-water equations,

$$\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + g\zeta = 0, \quad \nabla\cdot[(h+\zeta)\nabla\phi] + \zeta_t = 0, \qquad (2.1a, b)$$

for the velocity potential ϕ and the free-surface displacement ζ. Eliminating ζ, we obtain

$$g\nabla\cdot(h\nabla\phi)-\phi_{tt} = \nabla\phi\cdot\nabla\phi_t+\nabla\cdot\{[\phi_t+\tfrac{1}{2}(\nabla\phi)^2]\nabla\phi\}. \tag{2.2}$$

The boundary conditions are

$$h\phi_x \to 0 \ (x\downarrow 0), \quad \phi_y = 0 \quad (ky = 0 \bmod \pi), \tag{2.3a, b}$$

and (1.6) plus a radiation condition on the edge wave for $x\uparrow\infty$.

Starting from the hypothesis that the amplitude of the edge wave relative to that of the standing wave is $O(\epsilon^{-\frac{1}{2}})$ near the shore, we posit the perturbation expansion

$$\phi = \sigma g(k\omega)^{-1}[\epsilon^{\frac{1}{2}}\phi_1+\epsilon(\phi_0+\phi_2)+\epsilon^{\frac{3}{2}}\phi_3+O(\epsilon^2)], \tag{2.4}$$

where: the ϕ_n are dimensionless, $O(1)$ functions of the dimensionless variables

$$\xi \equiv kx, \quad \eta \equiv ky, \quad \theta \equiv \omega t, \quad \tau \equiv \epsilon\omega t; \tag{2.5}$$

ϕ_0 is the dimensionless counterpart of the standing wave ϕ_s; ϕ_1 is the counterpart of ϕ_e and an eigensolution of the linear edge-wave problem for $\lambda = \lambda_0$; ϕ_2 is driven by the quadratic self-interaction of ϕ_1; ϕ_3 is driven by the resonant detuning ($\propto \lambda-\lambda_0 \equiv 2\epsilon\beta\lambda$) and the slow ($\tau$) variation of ϕ_1, the quadratic interaction between ϕ_0 and ϕ_1, and the cubic self-interaction of ϕ_1. Substituting (2.4) and (2.5) into (2.2), invoking (1.2)–(1.4), and renormalizing $\nabla \equiv (\partial_\xi, \partial_\eta)$, we obtain

$$\nabla\cdot(\hbar\nabla\phi_0)-\lambda\phi_{0\theta\theta} \equiv \mathscr{L}\phi_0 = 0, \tag{2.6a}$$

$$\mathscr{L}\phi_1 = 0 \quad (\lambda = \lambda_0), \tag{2.6b}$$

$$\mathscr{L}\phi_2 = 2\nabla\phi_1\cdot\nabla\phi_{1\theta}+\phi_{1\theta}\nabla^2\phi_1, \tag{2.6c}$$

$$\begin{aligned}\mathscr{L}\phi_3 = {} & 2\lambda(\phi_{1\theta\tau}+\beta\phi_{1\theta\theta})+2(\nabla\phi_0\cdot\nabla\phi_1)_\theta+\phi_{0\theta}\nabla^2\phi_1+\phi_{1\theta}\nabla^2\phi_0 \\ & +2(\nabla\phi_1\cdot\nabla\phi_2)_\theta+\phi_{1\theta}\nabla^2\phi_2+\phi_{2\theta}\nabla^2\phi_1+\tfrac{1}{2}\lambda^{-1}\nabla\cdot(\nabla\phi_1)^3.\end{aligned} \tag{2.6d}$$

We note that, here and subsequently, $\lambda-\lambda_0$ may be neglected except in the tuning parameter β, and we attach the subscript to λ_0 only where necessary to identify it as an eigenvalue.

The boundary conditions implied by (1.5)–(1.7), (2.3) and the edge-wave radiation condition are

$$\phi_0 \to \tfrac{1}{2}\cos 2\theta, \quad \phi_1 \to (-p\sin\theta+q\cos\theta)\cos\eta, \quad \hbar\phi_{2\xi}\to 0 \quad (\xi\downarrow 0), \tag{2.7a–c}$$

$$\phi_0 \sim \tfrac{1}{2}A\cos(\hbar\xi-k_\infty l)\cos 2\theta, \quad \phi_1 \to 0, \quad \phi_{2\xi}+\tfrac{1}{2}\hbar\phi_{2\theta}\to 0 \quad (\xi\uparrow\infty), \tag{2.8a–c}$$

where

$$A \equiv a_\infty/a, \quad \hbar \equiv k_\infty/k = 2(\lambda\mu)^{\frac{1}{2}}. \tag{2.9a, b}$$

The boundary conditions (2.7a, b) incorporate the regularity implicit in (2.3a) and the normalizations implicit in (1.5) and (1.7).

3. Separation of variables

We pose the solutions of (2.6a, b), (2.7a, b) and (2.8a, b) in the forms

$$\phi_0 = \tfrac{1}{2}Z(\xi,\lambda)\cos 2\theta, \quad \phi_1 = (-p\sin\theta+q\cos\theta)E(\xi)\cos\eta, \tag{3.1a, b}$$

where Z and E (which give the offshore variations of ζ_s and ζ_e) are determined by

$$(\hbar Z')' + 4\lambda Z = 0, \quad (\hbar E')' + (\lambda_0 - \hbar) E = 0, \tag{3.2a, b}$$

$$Z = 1, \quad E = 1 \quad (\xi = 0), \tag{3.3a, b}$$

$$Z \sim A \cos(\hbar\xi - k_\infty l), \quad E \to 0 \quad (\xi \uparrow \infty), \tag{3.4a, b}$$

and, here and subsequently, $' \equiv \mathrm{d}/\mathrm{d}\xi$ except as noted. We display λ in $Z = Z(\xi, \lambda)$ in anticipation of its role as a spectral variable.

Substituting (3.1*b*) into (2.6*c*), we obtain

$$\mathscr{L}\phi_2 = (p^2+q^2)\{F(\xi) + F_2(\xi)\cos 2\eta\}\sin 2\hat{\theta}, \tag{3.5}$$

where
$$F \equiv \tfrac{1}{4}(E^2 + 2E'^2 + EE''), \quad F_2 \equiv F - E^2, \tag{3.6a, b}$$

and
$$\hat{\theta} \equiv \theta - \tan^{-1}(q/p). \tag{3.7}$$

Posing
$$\phi_2 = (p^2+q^2)\,\mathrm{Re}\{\mathrm{i}[\Phi(\xi) + \Phi_2(\xi)\cos 2\eta]\,\mathrm{e}^{-2\mathrm{i}\hat{\theta}}\}, \tag{3.8}$$

where Re implies the real part of, in (3.5), we obtain

$$(\hbar\Phi')' + 4\lambda\Phi = F, \quad (\hbar\Phi_2')' + 4(\lambda - \hbar)\Phi_2 = F_2. \tag{3.9a, b}$$

The boundary conditions (2.7*c*) and (2.8*c*) imply

$$\hbar\Phi' \to 0 \ (\xi \downarrow 0), \quad \Phi' - \mathrm{i}\hbar\Phi \to 0 \quad (\xi \uparrow \infty), \tag{3.10a, b}$$

for both Φ and Φ_2 (which may either radiate or decay exponentially as $\xi \uparrow \infty$).

Invoking $\hbar = 0$ and $\hbar' = 1$ at $\xi = 0$ in (3.2) and (3.9), we obtain the auxiliary conditions

$$Z' + 4\lambda Z = 0, \quad E' + \lambda_0 E = 0, \quad \Phi' + 4\lambda\Phi = F, \quad \Phi_2' + 4\lambda\Phi_2 = F_2 \quad (\xi = 0). \tag{3.11a–d}$$

The spectrum of (3.9*a*) is continuous over $0 < \lambda < \infty$, and the required solution may be constructed using Sturm–Liouville theory with the re-normalized solution

$$\hat{Z}(\xi, \lambda) \equiv (2\pi)^{-\frac{1}{2}}(\mu/\lambda)^{\frac{1}{4}} A^{-1} Z(\xi, \lambda) \tag{3.12}$$

of (3.2*a*) as the basis for a Fourier-integral representation (cf. Morse & Feshbach 1953). The end result is (cf. Minzoni & Whitham 1977)

$$\Phi = \int_0^\infty F(\eta)\,\mathrm{d}\eta \int_0^\infty \frac{\hat{Z}(\xi,\kappa)\hat{Z}(\eta,\kappa)\,\mathrm{d}\kappa}{\lambda - \kappa} \tag{3.13a}$$

$$= ⨍_0^\infty \frac{\mathscr{F}(\kappa)\hat{Z}(\xi,\kappa)\,\mathrm{d}\kappa}{\lambda-\kappa} - \mathrm{i}\pi\mathscr{F}(\lambda)\hat{Z}(\xi,\lambda), \tag{3.13b}$$

where
$$\mathscr{F}(\kappa) \equiv \int_0^\infty F(\xi)\hat{Z}(\xi,\kappa)\,\mathrm{d}\xi \tag{3.14}$$

is the Fourier transform of F, the integral in (3.13*a*) is indented under the pole at $\kappa = \lambda$ in order to satisfy the radiation condition (3.10*b*), and the crossed integral sign in (3.13*b*) implies a Cauchy principal value. We note that

$$\int_0^\infty F\Phi\,\mathrm{d}\xi = ⨍_0^\infty \frac{\mathscr{F}^2(\kappa)\,\mathrm{d}\kappa}{\lambda-\kappa} - \mathrm{i}\pi\mathscr{F}^2(\lambda). \tag{3.15}$$

The spectrum of (3.9*b*) comprises both discrete and continuous components, corresponding, respectively, to the $N+1$ discrete eigenvalues $0 < \lambda_{20} < \ldots < \lambda_{2N}$ and

$\lambda > k_\infty = 1/\mu$. We examine the complete spectrum and obtain explicit results for the special case of an exponential depth profile in Appendix A. These results suggest (but I have not proved) that, for any smooth profile, $N = 0$ for $\mu \leqslant \mu_1, \mu_1 \gtrsim 1$, that the contribution of the continuous spectrum to Φ_2 is exponentially (in $1/\mu$) small as $\mu \downarrow 0$, and that the single-mode truncation

$$\Phi_2 = \tfrac{1}{4}(\lambda-\lambda_{20})^{-1}\left(\int_0^\infty F_2\,\Phi_{20}\,\mathrm{d}\xi \Big/ \int_0^\infty \Phi_{20}^2\,\mathrm{d}\xi\right)\Phi_{20}(\xi), \tag{3.16}$$

where λ_{20} and Φ_{20} are the dominant eigenvalue and eigensolution of (3.9*b*), provides a good approximation for $\mu \lesssim 1$. Comparing (3.2*b*) and (3.9*b*), we infer that λ_{20} and Φ_{20} may be determined from λ_0 and E through a scale transformation ($\omega \to 2\omega$, $k \to 2k$).

4. Evolution equations

The evolution equations for p and q are determined by the solvability conditions that $\mathscr{L}\phi_3$ in (2.6*d*) be orthogonal to the linearly independent eigenfunctions $E(\xi)\cos\eta\cos\theta$ and $E(\xi)\cos\eta\sin\theta$ of (2.6*b*). Substituting (3.1) and (3.8) into (2.6*d*), invoking these orthogonality conditions (the procedure is lengthy but standard and includes the reduction of the various integrals through Green's theorem and the invocation of (3.3), (3.10*a*) and (3.11) at $\xi = 0$), solving for $\dot{p} \equiv \mathrm{d}p/\mathrm{d}\tau$ and $\dot{q}$, and incorporating boundary-layer damping but neglecting $Q = O(\alpha\epsilon)$ (see §7), we obtain (cf. (1.8))

$$\dot{p} = -[\alpha+P+R(p^2+q^2)]\,p-[\beta+S(p^2+q^2)]\,q, \tag{4.1a}$$

$$\dot{q} = -[\alpha-P+R(p^2+q^2)]\,q+[\beta+S(p^2+q^2)]\,p, \tag{4.1b}$$

or, equivalently, after introducing $r \equiv p+\mathrm{i}q$ and $r^* \equiv p-\mathrm{i}q$,

$$\dot{r} = (-\alpha+\mathrm{i}\beta)\,r-Pr^*+(-R+\mathrm{i}S)\,r^2r^*, \tag{4.2}$$

where
$$P = (\lambda[E^2])^{-1}[FZ] \tag{4.3a}$$

and
$$S+\mathrm{i}R = (2\lambda[E^2])^{-1}\left\{-4[F\Phi+\tfrac{1}{2}F_2\,\Phi_2]+\frac{3}{32\lambda}[3E'^4+2E^2E'^2+3E^4]+\left(\frac{9\lambda^2-5}{32}\right)\right\}, \tag{4.3b}$$

wherein
$$[f] \equiv \int_0^\infty f\,\mathrm{d}\xi.$$

It follows from (3.15) and (4.3*b*) that $R > 0$.

We now assume that $S < 0$; the following results may be continued into $S > 0$ by changing the signs of β and q. The fixed points of (4.1), at which $\dot{p} = \dot{q} = 0$, are given by

$$p = q = 0, \tag{4.4a}$$

$$p^2+q^2 = r_+^2 \quad (0 < \alpha < P, \quad -\beta_1 < \beta < \beta_1), \tag{4.4b}$$

$$p^2+q^2 = r_\pm^2, \quad (0 < \alpha < \alpha_*, \quad \beta_1 < \beta < \beta_*), \tag{4.4c}$$

where

$$\alpha_* = (R^2+S^2)^{-\frac{1}{2}}|S|\,P, \quad \beta_* = R^{-1}[P(R^2+S^2)^{\frac{1}{2}}-|S|\,\alpha], \quad \beta_1 = (P^2-\alpha^2)^{\frac{1}{2}}, \tag{4.5a–c}$$

$$r_\pm^2 = \frac{-R\alpha+|S|\,\beta\pm[P^2(R^2+S^2)-(R\beta+|S|\,\alpha)^2]^{\frac{1}{2}}}{R^2+S^2}, \tag{4.6}$$

and $$\frac{q}{p}=\frac{|S|\,r^2-\beta}{P-\alpha-Rr^2} \quad (r=r_\pm), \tag{4.7}$$

(there are two fixed points for each of $p^2+q^2=r_+^2$ and r_-^2). The origin is the only fixed point, and edge waves are not excited, if $\alpha > P$. There are symmetry-breaking (pitchfork) bifurcations at $\beta = \pm\beta_1$ (see below), and (4.4*b*) is admissible, if and only if $\alpha < P$. There is a turning-point (saddle-node) bifurcation at $\beta=\beta_* > \beta_1$, and (4.4*c*) is admissible, if and only if $\alpha < \alpha_*$.

The stability of a particular fixed point, say (p_0, q_0), may be determined by substituting
$$(p,q)=(p_0,q_0)+(p_1,q_1)\,\mathrm{e}^{\lambda\tau} \tag{4.8}$$
into (4.1), neglecting terms of second or higher order in (p_1,q_1), and requiring the determinant, $\Delta(\lambda)$, of the resulting linear equations in (p_1,q_1) to vanish. The result for $p_0=q_0=0$ is
$$\Delta(\lambda)=\lambda^2+2\alpha\lambda+\alpha^2+\beta^2-P^2=0, \tag{4.9}$$
from which it follows that: $p=q=0$ is stable/unstable for $\beta^2 \gtrless \beta_1^2$. The corresponding analyses for the fixed points (4.4*b*, *c*) reveal that $p^2+q^2=r_+^2$ is stable for $\beta^2<\beta_1^2$ if $\alpha_*<\alpha<P$ or for $-\beta_1<\beta<\beta_*$ if $0<\alpha<\alpha_*$, and that $p^2+q^2=r_-^2$ is unstable for $\beta_1<\beta<\beta_*$ if $0<\alpha<\alpha_*$. The stable/unstable fixed points are sinks/saddle points.

A more detailed analysis of the bifurcation points at $\beta=\pm\beta$, is expedited by the canonical transformation
$$\begin{bmatrix} p \\ q \end{bmatrix}=\begin{bmatrix} A & B \\ B & A \end{bmatrix}\begin{bmatrix} \hat{p} \\ \hat{q} \end{bmatrix}, \tag{4.10}$$
where $$A=\left(\frac{P+\alpha_1}{2\alpha_1}\right)^{\frac{1}{2}},\quad B=-\left(\frac{P-\alpha_1}{2\alpha_1}\right)^{\frac{1}{2}},\quad \alpha_1=(P^2-\beta^2)^{\frac{1}{2}}. \tag{4.11 a–c}$$

Transforming (1.9) with $Q=0$ in (1.10*b*), we obtain
$$[\partial_\tau+\alpha+R(p^2+q^2)]\begin{bmatrix} \hat{p} \\ \hat{q} \end{bmatrix}=\begin{bmatrix} -\partial H/\partial\hat{q} \\ \partial H/\partial\hat{p} \end{bmatrix} \tag{4.12a}$$
$$=\left\{\alpha_1\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}+\alpha_1^{-1}S(p^2+q^2)\begin{bmatrix} \beta & -P \\ P & -\beta \end{bmatrix}\right\}\begin{bmatrix} \hat{p} \\ \hat{q} \end{bmatrix}. \tag{4.12b}$$

It follows from the linear components of (4.12) that, in the neighbourhood of $\hat{p}=\hat{q}=0$, $\hat{q}$ grows like $\exp[(\alpha_1-\alpha)\tau]$, whereas $\hat{p}$ decays like $\exp[-(\alpha_1+\alpha)\tau]$. A centre-manifold projection (Guckenheimer & Holmes 1983, §3.2) of (4.12) yields
$$\partial_\tau\hat{q}=C_1\hat{q}-C_3\hat{q}^3+O(\hat{q}^5), \tag{4.13}$$
where $$C_1=\alpha_1-\alpha=\frac{\beta_1^2-\beta^2}{\alpha+\alpha_1} \tag{4.14a}$$
and $$C_3=\frac{P(R\alpha_1-|S|\,\beta)}{\alpha_1^2}=\frac{P(R^2+S^2)(\alpha^2-\alpha_*^2+\beta_1^2-\beta^2)}{\alpha_1^2(R\alpha_1+|S|\,\beta)}. \tag{4.14b}$$

It follows from (4.13) and (4.14) that there are pitchfork bifurcations at $\beta=\pm\beta_1$, that $\beta=-\beta_1$ is supercritical (since $C_3>0$ at $\beta=-\beta_1$), and that $\beta=\beta_1$ is super/subcritical for $\alpha \gtrless \alpha_*$.

The preceding results assume perfect reflection, for which $Q=0$. Allowing for imperfect reflection (see §7), we find that: $-Qq$, $-Qp$ and $-\mathrm{i}Qr^*$ are added to the

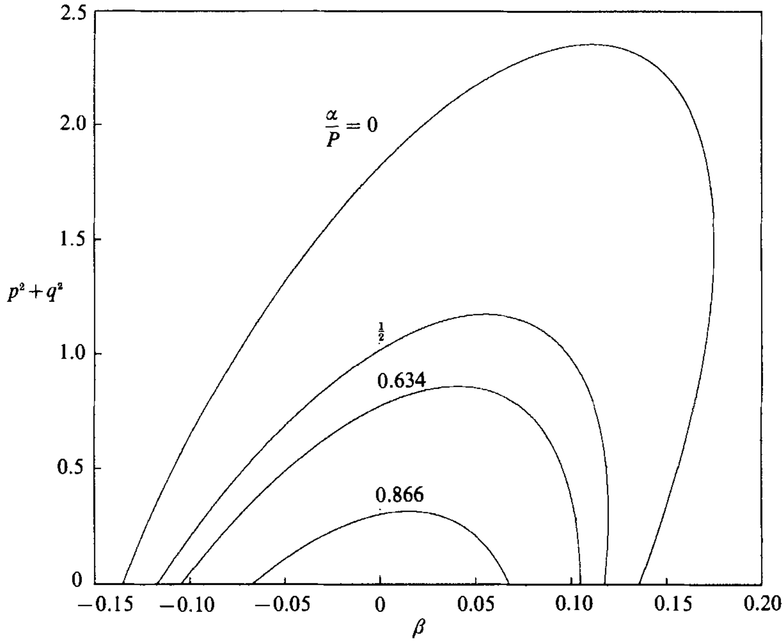

FIGURE 1. Resonance curves calculated from (4.6) for the limiting values $(\mu \downarrow 0)$ $P = 0.135$, $R = 0.0575$ and $S = -0.0471$.

right-hand sides of (4.1*a*), (4.1*b*) and (4.2), respectively; P is replaced by $(P^2+Q^2)^{\frac{1}{2}}$ in (4.4)–(4.6); Q is added to the numerator on the right-hand side of (4.7). This implies an increase in both the range of damping for which, and the bandwidth in which, edge waves may be excited.

The Poincaré–Bendixson theorem implies that any solution of (4.1) must tend asymptotically to either a fixed point or a limit cycle. The logarithmic contraction rate for the area within a closed orbit in the (p, q)-plane is given by

$$\frac{\partial \dot{p}}{\partial p}+\frac{\partial \dot{q}}{\partial q} = -2\alpha - 4R(p^2+q^2) < 0, \tag{4.15}$$

in consequence of which limit cycles are impossible and every solution of (4.1) must tend to one of the stable fixed points.

The resonance curves (dimensionless energy, p^2+q^2, *vs.* frequency offset, β), which have maxima at $\beta = |S|(P-\alpha)/R$ and $r_+^2 = (P-\alpha)/R$, are plotted for the asymptotic $(\mu \downarrow 0)$ values of P, R and S (see §5) in figure 1 (cf. Guza & Bowen 1976, figure 4; Rockliff 1978, figure 1). The damping coefficient for laminar damping, as calculated from (7.4) below with $\tan^{-1}\sigma = 5.1°$, $k = 2\pi/1.62$ m, $a = 1.15$ cm (data cited by Guza & Bowen for $\epsilon_l = 1$ in their notation), $\nu = 10^{-2}$ cm²/s and $C = 2$, is $\alpha = 0.090$, which gives $\alpha/P = 0.67$ (cf. $\alpha_*/P = 0.63$).

5. Exponential profile

The shallow-water equations admit exact solutions for

$$\hat{h} = \mu h = 1 - e^{-\mu\xi}. \tag{5.1}$$

Substituting (5.1) into (3.2*a*) and adopting $\exp(-\mu\xi)$ as the independent variable, we obtain (Miles 1990*b*)

$$Z(\xi,\lambda) = \mathrm{Re}\{{}_2F_1(1-\mathrm{i}\nu,\,-\mathrm{i}\nu;1;1-\mathrm{e}^{-\mu\xi})\,\mathrm{e}^{\frac{1}{2}\mathrm{i}\mu\nu\xi}\}, \tag{5.2a}$$

$$= \mathrm{Re}\{\mathscr{A}(\nu)\,{}_2F_1(1-\mathrm{i}\nu,\,-\mathrm{i}\nu;1-2\mathrm{i}\nu;\mathrm{e}^{-\mu\xi})\,\mathrm{e}^{\frac{1}{2}\mathrm{i}\mu\nu\xi}\}, \tag{5.2b}$$

where
$$\nu = 2\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}}, \quad \mathscr{A}(\nu) \equiv \frac{2\Gamma(2\mathrm{i}\nu)}{\Gamma(\mathrm{i}\nu)\,\Gamma(1+\mathrm{i}\nu)}, \tag{5.3a, b}$$

and ${}_2F_1$ is Gauss's hypergeometric function. Letting $\xi\uparrow\infty$ in (5.2*b*) and invoking (3.4*a*), we obtain

$$A = \frac{a_\infty}{a} = |\mathscr{A}| = \left(\frac{\tanh\pi\nu}{\pi\nu}\right)^{\frac{1}{2}}, \quad k_\infty l \bmod 2\pi = -\arg\mathscr{A} = \tfrac{1}{4}\pi - \nu\ln 4 - \tfrac{1}{8}\nu^{-1} + O(\nu^{-3}). \tag{5.4a, b}$$

Substituting (5.4*a*) into (3.12), we obtain

$$\hat{Z}(\xi,\lambda) = Z(\xi,\lambda)\,(\coth\pi\nu)^{\frac{1}{2}}. \tag{5.5}$$

The eigensolutions of (3.2*b*) are given by (Ball 1967)

$$E = E_n(\xi,\mu) = {}_2F_1(-n,\,2\mu^{-1}\lambda_0-n+1;1;1-\mathrm{e}^{-\mu\xi})\,\mathrm{e}^{-(\lambda_0-n\mu)\xi}, \tag{5.6a}$$

$$\lambda_n = \lambda_n(\mu) = (2n+1)(1+\tfrac{1}{4}\mu^2)^{\frac{1}{2}} - (n^2+n+\tfrac{1}{2})\mu \quad (n = 0,1,\ldots,N), \tag{5.6b}$$

where the E_n are hypergeometric (but not Jacobi) polynomials, and N is the integral part of λ_0/μ. The dominant mode corresponds to $n=0$, for which

$$E_0 = \mathrm{e}^{-\lambda_0\xi}, \quad \lambda_0 = (1+\tfrac{1}{4}\mu^2)^{\frac{1}{2}} - \tfrac{1}{2}\mu. \tag{5.7a, b}$$

The solutions of (3.9*b*) are determined in Appendix A. The dominant mode, which is the only member of the discrete spectrum if $\mu<\sqrt{2}$, is given by

$$\Phi_{20} = \mathrm{e}^{-4\lambda_{20}\xi}, \quad \lambda_{20} = \tfrac{1}{2}(1+\tfrac{1}{16}\mu^2)^{\frac{1}{2}} - \tfrac{1}{8}\mu. \tag{5.8a, b}$$

Substituting (5.7*a*) into (3.6*a*, *b*) and invoking $\lambda\simeq\lambda_0$, we obtain

$$F = \tfrac{1}{4}(3\lambda^2+1)\,\mathrm{e}^{-2\lambda\xi}, \quad F_2 = \tfrac{3}{4}(\lambda^2-1)\,\mathrm{e}^{-2\lambda\xi}. \tag{5.9a, b}$$

Combining (5.2*a*), (5.7*a*) and (5.9*a*) in (4.3*a*), we obtain

$$P(\lambda) = \tfrac{1}{2}(3\lambda^2+1)\int_0^\infty \mathrm{e}^{-2\lambda\xi}Z(\xi,\lambda)\,\mathrm{d}\xi \tag{5.10a}$$

$$= \tfrac{1}{2}\mu^{-1}(3\lambda^2+1)\,\mathrm{Re}\int_0^1 (1-z)^{\frac{1}{2}\nu^2-\mathrm{i}\nu-1}\,{}_2F_1(1-\mathrm{i}\nu,\,-\mathrm{i}\nu;1;z)\,\mathrm{d}z \tag{5.10b}$$

$$= \left(\frac{3\lambda^2+1}{4\lambda}\right)\left|\frac{\Gamma(\frac{1}{2}\nu^2+\mathrm{i}\nu)}{\Gamma(\frac{1}{2}\nu^2)}\right|^2, \tag{5.10c}$$

which is plotted in figure 2. Combining (5.5), (5.7*a*), (5.8*a*), (5.9*a*, *b*) and the approximation (3.16) in (4.3*b*), we obtain

$$R = 4\pi\mathscr{F}^2(\lambda) = \pi P^2\coth\pi\nu, \tag{5.11}$$

$$S = \frac{45\lambda^2-14+9\lambda^{-2}}{128} + 4\int_0^\infty \frac{\mathscr{F}^2(\kappa)\,\mathrm{d}\kappa}{\kappa-\lambda} - \frac{9}{16}\left(\frac{\lambda^2-1}{\lambda+2\lambda_{20}}\right)^2\left(\frac{\lambda_{20}}{\lambda-\lambda_{20}}\right), \tag{5.12}$$

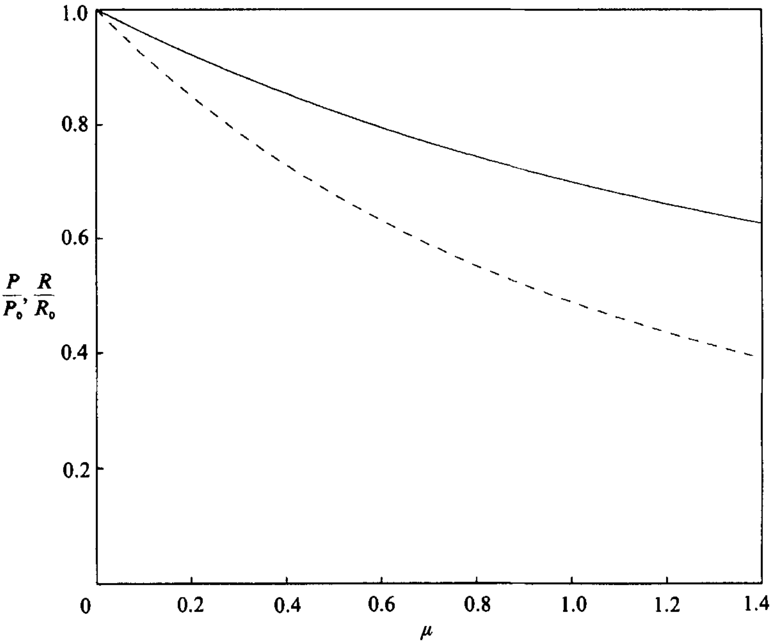

FIGURE 2. The parameters P/P_0 (——) and R/R_0 (---) for the exponential profile (5.1), as calculated from (5.10) and (5.11). $P_0 = e^{-2}$ and $R_0 = \pi e^{-4}$.

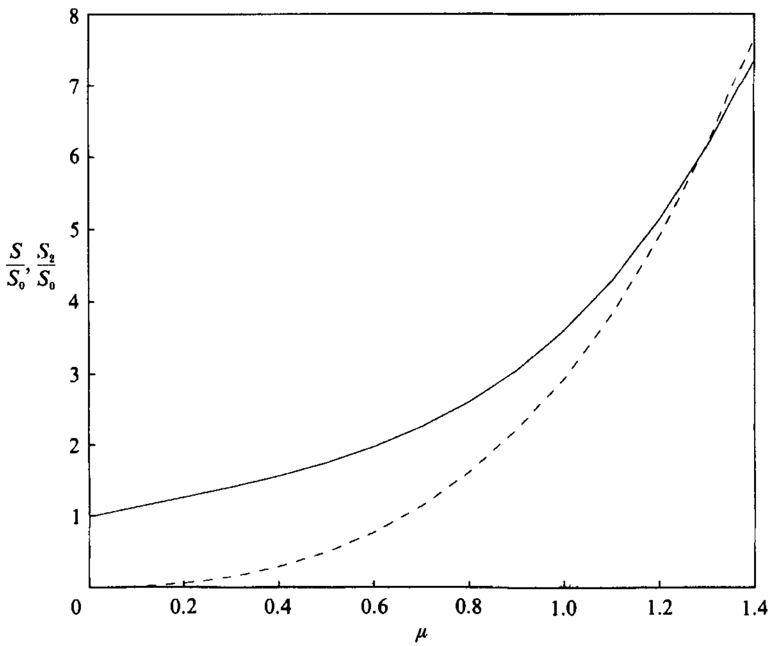

FIGURE 3. The parameters S/S_0 (——) and S_2/S_0 (---) for the exponential profile, as calculated from (5.12) (in which S_2 is the last term). $S_0 = -0.0471$.

where $$\mathscr{F}(\kappa) = \left(\frac{3\lambda^2+1}{8\lambda}\right)\left|\frac{\Gamma(\frac{1}{2}\nu^2+\mathrm{i}\nu_\kappa)}{\Gamma(\frac{1}{2}\nu^2)}\right|^2[\coth(\pi\nu_\kappa)]^{\frac{1}{2}}, \quad \nu_\kappa \equiv 2\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}. \qquad (5.13a, b)$$

The integral in (5.12) is analytically intractable but has been evaluated numerically to obtain the results plotted in figure 3. Note that S_2 (the last term in (5.12)) is negligible as $\mu \downarrow 0$ but dominates S for $\mu \geqslant 0.7$.

The asymptotic development of the preceding results as $\mu \downarrow 0$ yields (Appendix B)

$$P = \mathrm{e}^{-2}(1-\tfrac{5}{12}\mu+\tfrac{47}{360}\mu^2), \quad R = \pi\,\mathrm{e}^{-4}(1-\tfrac{5}{6}\mu+\tfrac{313}{720}\mu^2), \qquad (5.14a, b)$$

$$S = \tfrac{5}{16}-\mathscr{E}-(\tfrac{35}{16}-5\mathscr{E})\tfrac{1}{6}\mu+(\tfrac{569}{8}-313\mathscr{E})\tfrac{1}{720}\mu^2 = -0.04705(1+1.3806\mu+1.2225\mu^2), \qquad (5.14c)$$

wherein $O(\mu^3)$ errors are implicit and $\mathscr{E} \equiv \mathrm{e}^{-4}\,\mathrm{Ei}(4)$. The errors in (5.14*a*) and (5.14*b*) are less than 2 and 4%, respectively, for $\mu \leqslant 1$. The error in (5.14*c*) is less than 5% for $\mu \leqslant 0.7$.

6. Asymptotic ($\mu \downarrow 0$) approximations

Posing the solution of (3.2*a*) in the form

$$Z = \mathscr{k}^{-\frac{1}{4}}(\tfrac{1}{2}\chi)^{\frac{1}{2}}f(\chi), \quad \chi = \int_0^{\xi} \mathscr{k}^{-\frac{1}{2}}\,\mathrm{d}\xi, \qquad (6.1a, b)$$

solving the resulting equation for f by expanding in powers of μ, and invoking (3.3*a*), which implies $f(0) = 1$, we obtain (Miles 1990*b*)

$$f(\chi) = J_0[(4\lambda-r)^{\frac{1}{2}}\chi]+O(\mu^2), \quad r = \tfrac{1}{4}(\mathscr{k}''-\tfrac{1}{4}\mathscr{k}^{-1}\mathscr{k}'^2+\chi^{-2}), \qquad (6.2a, b)$$

where J_0 is a Bessel function, $r = O(\mu)$, and the approximation is uniformly valid in $0 < \chi < \infty$ as $\mu \downarrow 0$ with $\lambda = O(1)$.

The inner approximation (which suffices for the present calculation) to the dominant eigensolution of (3.2*b*), (3.3*b*) and (3.4*b*) is given by (Miles 1989)

$$E^{(i)} = \mathrm{e}^{-\lambda_0\xi}[1+O(\mu^2)], \quad \lambda_0 = 1+\tfrac{1}{2}\gamma+O(\mu^2), \quad \gamma \equiv \frac{(\mathrm{d}^2h/\mathrm{d}x^2)_0}{\sigma k} = O(\mu). \quad (6.3a\text{–}c)$$

Substituting (6.3*a*) into (3.6) and invoking $\lambda_0 \simeq \lambda$, we obtain the inner approximations (cf. (5.9))

$$F = \tfrac{1}{4}(3\lambda^2+1)\,\mathrm{e}^{-2\lambda\xi}, \quad F_2 = \tfrac{3}{4}(\lambda^2-1)\,\mathrm{e}^{-2\lambda\xi}. \qquad (6.4a, b)$$

It follows from (6.3*b*) and (6.4*b*) that F_2, and hence also Φ_2, is $O(\mu)$, and hence that $[F_2\,\Phi_2] = O(\mu^2)$ in (4.3*b*).

The asymptotic solution described by (6.1) and (6.2) requires $\lambda = O(1)$ as $\mu \downarrow 0$ and therefore is unsuitable for the determination of Φ from (3.13), which requires $\hat{Z}(\xi, \kappa)$ near $\kappa = 0$. Accordingly, we attack (3.9*a*) by posing (cf. (6.1))

$$\Phi = \mathscr{k}^{-\frac{1}{4}}(\tfrac{1}{2}\chi)^{\frac{1}{2}}\hat{\Phi}(\chi), \quad F = \mathscr{k}^{-\frac{1}{4}}(\tfrac{1}{2}\chi)^{\frac{1}{2}}\hat{F}(\chi), \qquad (6.5a, b)$$

where χ is given by (6.1*b*), to obtain

$$\hat{\Phi}''+\chi^{-1}\hat{\Phi}'+(4\lambda-r)\,\hat{\Phi} = \hat{F}. \qquad (6.6)$$

Solving (6.6) by variation of parameters and invoking (3.10*a, b*), we obtain

$$\hat{\Phi} = \tfrac{1}{2}\pi\left[Y(\chi)\int_0^{\chi}\hat{F}(\bar{\chi})\,J(\bar{\chi})\,\bar{\chi}\,\mathrm{d}\bar{\chi}+J(\chi)\int_{\chi}^{\infty}\hat{F}(\bar{\chi})\,Y(\bar{\chi})\,\bar{\chi}\,\mathrm{d}\bar{\chi}-\mathrm{i}J(\chi)\int_0^{\infty}\hat{F}(\bar{\chi})\,J(\bar{\chi})\,\bar{\chi}\,\mathrm{d}\bar{\chi}\right], \qquad (6.7a, b)$$

where $$J(\chi) \equiv J_0[(4\lambda - r)^{\frac{1}{2}}\chi], \quad Y(\chi) \equiv Y_0[4\lambda - r)^{\frac{1}{2}}\chi]. \tag{6.8a, b}$$

We require the integral

$$[F\Phi] \equiv \int_0^\infty F\Phi \,\mathrm{d}\xi = \pi \int_0^\infty FJ\hbar^{\frac{1}{4}}\chi^{\frac{1}{2}}\,\mathrm{d}\chi \int_\chi^\infty FY\hbar^{\frac{1}{4}}\bar{\chi}^{\frac{1}{2}}\,\mathrm{d}\bar{\chi} - \tfrac{1}{2}\mathrm{i}\pi\left(\int_0^\infty FJ\hbar^{\frac{1}{4}}\chi^{\frac{1}{2}}\,\mathrm{d}\chi\right)^2. \tag{6.9}$$

Combining (6.2a, b), (6.3a, b), (6.4a), (6.8a, b) and (6.9) in (4.3), neglecting $[F_2\,\Phi_2] = O(\mu^2)$, invoking the inner approximations

$$\hbar = \xi + \tfrac{1}{2}\gamma\xi^2, \quad \chi = 2\xi^{\frac{1}{2}} - \tfrac{1}{6}\gamma\xi^{\frac{3}{2}}, \quad r = \tfrac{1}{6}\gamma, \tag{6.10a–c}$$

where γ is defined by (6.3c), introducing $s \equiv (\lambda - \frac{1}{4}r)^{\frac{1}{2}}\chi$ as the variable of integration, and expanding in powers of $\gamma = O(\mu)$, we obtain

$$P = \int_0^\infty \mathrm{e}^{-\frac{1}{2}s^2} J_0(2s)\,p(s)\,\mathrm{d}s, \quad R = \pi P^2, \tag{6.11a, b}$$

and $$S = \tfrac{5}{16} + \tfrac{9}{32}\gamma - 2\pi \int_0^\infty \mathrm{e}^{-\frac{1}{2}s^2} J_0(2s)\,p(s)\,\mathrm{d}s \int_s^\infty \mathrm{e}^{-\frac{1}{2}t^2} Y_0(2t)\,p(t)\,\mathrm{d}t, \tag{6.11c}$$

where $$p(s) = s\left[1 + \gamma\left(\frac{14 + s^2 - s^4}{48}\right)\right] + O(\mu^2). \tag{6.12}$$

Evaluating P with the aid of Weber's integral

$$\int_0^\infty J_0(at)\exp(-bt^2)\,t\,\mathrm{d}t = \tfrac{1}{2}b^{-1}\exp(-\tfrac{1}{4}a^2b^{-1}) \tag{6.13}$$

(and its derivatives with respect to b) and the integral in (6.11c) numerically, we obtain

$$P = \mathrm{e}^{-2}(1 + \tfrac{5}{12}\gamma), \quad R = \pi\,\mathrm{e}^{-4}(1 + \tfrac{5}{6}\gamma), \quad S = -0.04705 + 0.06496\gamma, \tag{6.14a–c}$$

wherein $O(\gamma^2)$ errors are implicit. The results for $\gamma = 0$ agree with Minzoni & Whitham (1977) and Rockliff (1978); Guza & Bowen (1976) obtain $S = -0.055$. The $O(\gamma)$ terms agree with those in (5.14), for which $\gamma = -\mu$. (This agreement provides a mutual check on the relatively complicated, essentially independent calculations of $[F\Phi]$.) We remark that (6.14) are independent of kh_∞, which suggests that (for $|\gamma| \ll 1$) they are not subject to the shallow-water restriction $kh_\infty \ll 1$ (cf. Minzoni & Whitham (1977), who demonstrate this independence for $\gamma = 0$ and $\sigma \ll 1$).

7. Viscosity and capillarity

Viscosity requires the replacement of (3.1a, b) by

$$\phi_0 = \tfrac{1}{2}\mathrm{Re}\{Z(\xi)\,\mathrm{e}^{-2\mathrm{i}\theta}\}, \quad \phi_1 = \mathrm{Re}\{(p + \mathrm{i}q)\,E(\xi)\,\mathrm{e}^{-\mathrm{i}(\theta + \frac{1}{2}\pi)}\cos\eta\}, \tag{7.1a, b}$$

in which Z and E now are complex amplitudes. (Viscous modifications of ϕ_2 are negligible in the present approximation; in particular, the inviscid approximation to Z may be used in (3.12)–(3.14).) If capillarity is neglected Z and E satisfy second-order differential equations that have irregular singular points at $h = 0$. Capillarity raises the order of these equations to four and renders the singularity at $h = 0$ regular (Miles 1990a).

If $kl_*, k\delta_* \ll 1$, where l_* is the capillary length (2.8 mm for clean water),

$$\delta_* = (\nu/2\omega)^{\frac{1}{2}} \tag{7.2}$$

is a viscous lengthscale, and ν is the kinematic viscosity, the primary effect of viscosity on the edge wave is to render the resonant frequency complex. The end result for the Stokes edge wave is (Miles 1990*a*)

$$\omega_0 = (\sigma g k)^{\frac{1}{2}}[1 + kl_* - C(1+\mathrm{i})\,\sigma^{-1}k\delta_*] \equiv \omega_r + \mathrm{i}\omega_i, \tag{7.3}$$

where $C = 1$ for a clean free surface or $C = 2$ for a fully contaminated surface. The corresponding approximation to the linear damping parameter in (1.9*a*) and (4.1) is

$$\alpha = \epsilon^{-1}(-\omega_i/\omega_r) = C\delta_*/a, \tag{7.4}$$

wherein ϵ^{-1} ($\epsilon \equiv ka/\sigma$) allows for the scaling of the slow time ($\tau \equiv \epsilon\omega_r t$), and an error factor of $1 + O(kl_*, k\delta_*, \mu)$ is implicit. We remark that (7.4) also may be obtained through a boundary-layer approximation (cf. Guza & Bowen 1976) despite the violation of the boundary-layer assumption $\delta_* \ll h$ near $h = 0$.

The primary effect of viscosity on the basic wave represented by Z is to render the reflection imperfect, in consequence of which (6.1) and (6.2*a*) with $r = 0$ (in first approximation) therein are replaced by the outer approximation (Miles 1990*b*)

$$Z = (\hbar - \hbar_\delta)^{-\frac{1}{4}}(\tfrac{1}{2}\chi)^{\frac{1}{2}}f(\chi), \quad \chi = \int_{\xi_\delta}^{\xi} (\hbar - \hbar_\delta)^{-\frac{1}{2}}\,\mathrm{d}\xi, \tag{7.5a, b}$$

where
$$\hbar_\delta \equiv \hbar(\xi_\delta) = (1+\mathrm{i})\,C\sigma^{-1}k\delta_* = (1+\mathrm{i})\,\alpha\epsilon \tag{7.6}$$

and
$$f(\chi) = J_0(2\chi) - 4\pi(\hbar_\delta - kl_*)\,Y_0(2\chi) + O(\mu). \tag{7.7}$$

We remark that a in (1.5) now is the amplitude at that station at which $|Z| = 1$, rather than the amplitude at $x = 0$ (at which the inner approximation to Z vanishes). The present formulation may be renormalized by replacing the reference station $x = 0$ in (1.5) and (1.7) by $x = x_1$ and dividing (7.6*a*) by $(\hbar_1 - \hbar_\delta)^{-\frac{1}{4}}(\frac{1}{2}\chi_1)^{\frac{1}{2}}f(\chi_1)$.

Modifying the calculation described in the first paragraph of §4 to allow for the complexity of E and Z, we find that $-Qq$, $-Qp$ and $-\mathrm{i}Qr^*$ are added to the right-hand sides of (4.1*a*), (4.1*b*) and (4.2), respectively, while (4.3*a*) is replaced by

$$P + \mathrm{i}Q = (\lambda[|E|^2])^{-1}[\bar{F}Z], \tag{7.8}$$

where $\bar{F}$ is the complex conjugate of F, as defined by (3.6*a*) for complex E. Proceeding as in §6, neglecting $\hbar_\delta$ relative to $\hbar$ in (7.5), and approximating E by (6.3) (on the assumption that the contribution to Q of the imaginary part of E is dominated by that of Z), we obtain (cf. (6.11*a*))

$$P + \mathrm{i}Q = \int_0^\infty \mathrm{e}^{-\frac{1}{2}s^2}[J_0(2s) - 4\pi(\hbar_\delta - kl_*)\,Y_0(2s)]\,s\,\mathrm{d}s = \mathrm{e}^{-2}[1 - 4\mathrm{Ei}(2)\,(\hbar_\delta - kl_*)], \tag{7.9}$$

where Ei (2) = 4.954 is an exponential integral.† It follows from (7.6) and (7.9) that $Q = O(\alpha\epsilon)$ and therefore is negligible in the limit $\epsilon \downarrow 0$; however, it may not be numerically negligible for moderately small $\alpha\epsilon$.

The numerical calculations were carried out by Ms Janet Becker. This work was supported in part by the Physical Oceanography, Applied Mathematics and Fluid Dynamics/Hydraulics programmes of the National Science Foundation, NSF Grants OCE-81-17539 and OCE-85-18763, by the Office of Naval Research, Contract

† The result for $\int_0^\infty Y_\nu(at)\exp(-bt^2)\,t\,\mathrm{d}t$ that is given in standard tables (Erdélyi *et al.* 1954) for arbitrary ν is singular for $\nu = 0$, and I evaluated the integral in (7.9) by expanding Y_0 in s^n and $s^n \log s$ and integrating term by term.

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Appendix A. Determination of Φ_2 for exponential profile

The discrete eigenfunctions of (3.9*b*) for the exponential profile (5.1) are given by (cf. (5.6))

$$\Phi_{2n} = {}_2F_1(-n, 2\mathscr{N}-n+1;1;1-\mathrm{e}^{-\mu\xi})\,\mathrm{e}^{-(\mathscr{N}-n)\mu\xi}, \tag{A 1a}$$

and

$$\lambda_{2n} = (n+\tfrac{1}{2})[1+(\tfrac{1}{4}\mu)^2]^{\frac{1}{2}}-\tfrac{1}{4}(n^2+n+\tfrac{1}{2})\mu \quad (n=0,1,\ldots,N), \tag{A 1b}$$

where

$$\mathscr{N} \equiv (\tfrac{1}{4}+4\mu^{-2})^{\frac{1}{2}}-\tfrac{1}{2} = 4\mu^{-1}\lambda_{20}, \tag{A 1c}$$

and N is the integral part of $\mathscr{N}$; $N=0$ for $\mu<\surd 2$. We remark that $\lambda_{20}<\lambda_0$ and $\lambda_{2n}>\lambda_0\,(n\geqslant 1)$ in the admissible range of μ; accordingly, internal resonance between the dominant mode and any of the secondary modes is impossible.

The eigenfunction in the continuous spectrum, $1/\mu<\lambda<\infty$, is given by (cf. (5.2))

$$\Phi_{2\lambda}(\xi) = \mathrm{Re}\{{}_2F_1(1+\mathscr{N}-\mathrm{i}\nu,\,-\mathscr{N}-\mathrm{i}\nu;1;1-\mathrm{e}^{-\mu\xi})\,\mathrm{e}^{\mathrm{i}\mu\nu\xi}\} \tag{A 2a}$$

$$= \mathrm{Re}\{\mathscr{A}(\nu)\,{}_2F_1(1+\mathscr{N}-\mathrm{i}\nu,\,-\mathscr{N}-\mathrm{i}\nu;1-2\mathrm{i}\nu;\mathrm{e}^{-\mu\xi})\,\mathrm{e}^{\mathrm{i}\mu\nu\xi}\}, \tag{A 2b}$$

where

$$\nu = \frac{2(\lambda\mu-1)^{\frac{1}{2}}}{\mu},\quad \mathscr{A}(\nu) = \frac{2\Gamma(2\mathrm{i}\nu)}{\Gamma(1+\mathscr{N}+\mathrm{i}\nu)\,\Gamma(-\mathscr{N}+\mathrm{i}\nu)}. \tag{A 3a, b}$$

Substituting (5.9*b*) (which is now exact) into (3.9*b*) and expanding the solution in the eigenfunctions (A 1) and (A 2), we obtain (cf. (3.13))

$$\Phi_2 = \frac{1}{4}\sum_{n=0}^{N}\frac{\mathscr{F}_{2n}\,\Phi_{2n}(\xi)}{(\lambda-\lambda_{2n})\,\|\Phi_{2n}\|^2}+\int_{1/\mu}^{\infty}\frac{\mathscr{F}_2(\kappa)\,\hat{\Phi}_{2\kappa}(\xi)\,\mathrm{d}\kappa}{(\lambda-\kappa)}, \tag{A 4}$$

where

$$\mathscr{F}_{2n} = \int_0^\infty F_2(\xi)\,\Phi_{2n}(\xi)\,\mathrm{d}\xi = \frac{3(\lambda^2-1)\,\Gamma(2\mu^{-1}\lambda+\mathscr{N}-n)\,\Gamma(2\mu^{-1}\lambda-\mathscr{N}+n)}{4\mu\Gamma(2\mu^{-1}\lambda+\mathscr{N}+1)\,\Gamma(2\mu^{-1}\lambda-\mathscr{N})}, \tag{A 5a}$$

$$\|\Phi_{2n}\|^2 = \int_0^\infty \Phi_{2n}^2\,\mathrm{d}\xi = \mu^{-1}\int_0^1 (1-x)^{2\mathscr{N}-2n-1}\,{}_2F_1^2(-n,2\mathscr{N}-n+1;1;x)\,\mathrm{d}x, \tag{A 5b}$$

$$\hat{\Phi}_{2\lambda} = (\pi\nu)^{-\frac{1}{2}}A^{-1}\Phi_{2\lambda}, \tag{A 6a}$$

$$\mathscr{F}_2(\kappa) = \int_0^\infty F_2(\xi)\,\hat{\Phi}_{2\kappa}(\xi)\,\mathrm{d}\xi = \frac{3(\lambda^2-1)\,|\Gamma(2\mu^{-1}\lambda+\mathrm{i}\nu_\kappa)|^2}{4\mu\Gamma(2\mu^{-1}\lambda+\mathscr{N}+1)\,\Gamma(2\mu^{-1}\lambda-\mathscr{N})\,(\pi\nu_\kappa)^{\frac{1}{2}}A(\nu_\kappa)}, \tag{A 6b}$$

wherein ν_κ is given by (A 3*a*) with λ replaced by κ therein.

The contribution of Φ_2 to S (4.3*b*) is given by (after invoking $2\lambda[E^2]=1$

$$S_2 \equiv -2\int_0^\infty F_2\,\Phi_2\,\mathrm{d}\xi = -\frac{1}{2}\sum_{n=0}^{N}\frac{\mathscr{F}_{2n}^{\,2}}{(\lambda-\lambda_{2n})\,\|\Phi_{2n}\|^2}+2\int_{1/\mu}^{\infty}\frac{\mathscr{F}_2^{\,2}(\kappa)\,\mathrm{d}\kappa}{\kappa-\lambda}. \tag{A 7}$$

$N=0$ for $\mu<\surd 2$, and the integral is exponentially small as $\mu\downarrow 0$. The contributions of the dominant ($n=0$) mode and the continuous spectrum (obtained by numerical evaluation of the integral) for $\mu=\surd 2-$ are -0.433 and 0.007, and it appears that the error in retaining only the dominant mode in the calculation of S_2 is less than 1.6% for $\mu<\surd 2$.

Appendix B. Asymptotic calculation for exponential profile

The asymptotic development of the results in §5 follows from the power-series expansion

$$\lambda \simeq \lambda_0 = 1-\tfrac{1}{2}\mu+\tfrac{1}{8}\mu^2+O(\mu^3) \tag{B 1}$$

and the asymptotic expansion

$$\left|\frac{\Gamma(\frac{1}{2}\nu^2+\mathrm{i}\nu_\kappa)}{\Gamma(\frac{1}{2}\nu^2)}\right|^2 = \mathrm{e}^{-2\bar{\kappa}}\left[1+\frac{\mu}{\lambda}\left(-\frac{\bar{\kappa}}{2}+\frac{\bar{\kappa}^2}{3}\right)+\frac{\mu^2}{\lambda^2}\left(\frac{-\bar{\kappa}}{12}+\frac{3\bar{\kappa}^2}{8}-\frac{3\bar{\kappa}^3}{10}+\frac{\bar{\kappa}^4}{18}\right)+O(\mu^3)\right. \tag{B 2}$$

wherein $\bar{\kappa} \equiv \kappa/\lambda$. Combining (5.13), (B 1) and (B 2) in the integral in (5.12), separating the integral into two parts, (*a*) and (*b*), where $\coth(\pi\nu_\kappa)$ is replaced by 1 in (*a*) and by $\coth(\pi\nu_\kappa)-1$ in (*b*), and letting $\kappa = \lambda t$ in (*a*) and $\kappa \equiv \mu s^2$ in (*b*), we obtain

$$\begin{aligned}4\int_0^\infty \frac{\mathscr{F}^2(\kappa)\,\mathrm{d}\kappa}{\kappa-\lambda} &= (1-\tfrac{1}{2}\mu+\tfrac{5}{16}\mu^2)\int_0^\infty \frac{\mathrm{e}^{-4t}}{t-1}[1+\mu(-t+\tfrac{2}{3}t^2) \\ &\quad +\mu^2(-\tfrac{2}{3}t+\tfrac{4}{3}t^2-\tfrac{14}{15}t^3+\tfrac{2}{9}t^4)]\,\mathrm{d}t \\ &\quad -2\mu\int_0^\infty \mathrm{e}^{-2\pi s}\,\mathrm{cosech}\,(2\pi s)\,(s-3\mu s^3)\,\mathrm{d}s+O(\mu^3) \qquad (\mathrm{B}\ 3a) \\ &= (1-\tfrac{1}{2}\mu+\tfrac{5}{16}\mu^2)[-\mathrm{e}^{-4}\,\mathrm{Ei}\,(4)\,(1-\tfrac{1}{3}\mu-\tfrac{2}{45}\mu^2)-\tfrac{1}{24}\mu+\tfrac{31}{2880}\mu^2] \\ &\quad -2\mu[\tfrac{1}{48}+\tfrac{1}{640}\mu]+O(\mu^3), \qquad (\mathrm{B}\ 3b)\end{aligned}$$

where the integrals in (B 3*a*) have been evaluated from a table of Laplace transforms.

Substituting (B 1) and (B 2) with $\bar{\kappa} = 1$ therein into (5.10*c*) and invoking (5.11), we obtain (5.14*a, b*). Substituting (B 1), (B 3*b*) and

$$\lambda_{20} = \tfrac{1}{2}-\tfrac{1}{8}\mu+\tfrac{1}{64}\mu^2+O(\mu^3) \tag{B 4}$$

into (5.12), we obtain (5.14*c*). We remark that if the asymptotic ($\mu \downarrow 0$) approximation to $F(\kappa)$ had been used $\coth(\pi\nu_\kappa)$ in (5.13*a*) would have been approximated by unity, in consequence of which the second integral in (B 3*a*) would have been neglected and the resulting approximation to S would have been in error at $O(\mu)$.

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